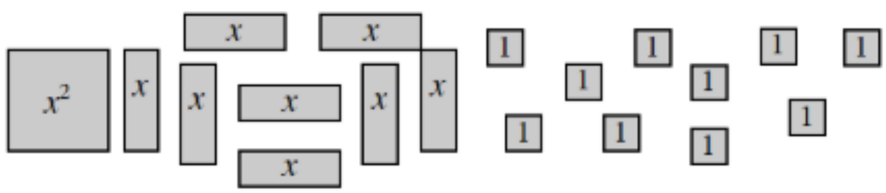
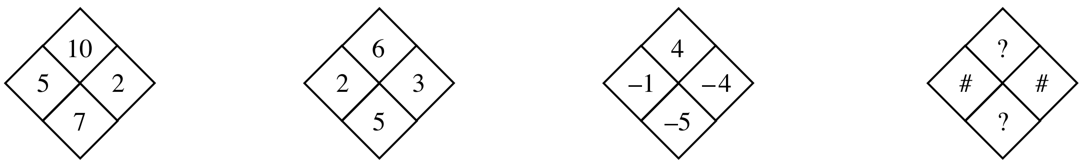
Using an Area Model to Teach Multiplying, Factoring, and Dividing Polynomials



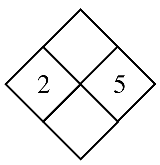
Getting Ready

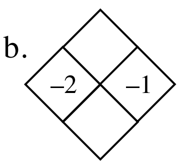
1. Finding and using a pattern is an important problem-solving skill you will use in algebra. The patterns in Diamond Problems will be used later to solve other types of algebraic problems.

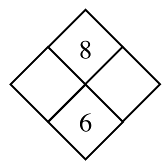
   Look for a pattern in the first three diamonds below. For the fourth diamond, explain how you could find the missing numbers (?) if you know the two numbers (#).

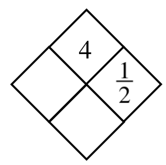


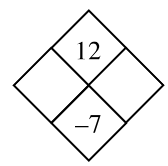
   Copy the Diamond Problems below. Then use the pattern you discovered to complete each one.

a. 

b. 

c. 

d. 

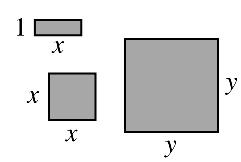
e. 

https://technology.cpm.org/general/tiles/

2. Your teacher will distribute a set of algebra tiles for your team to use.

   a. The tiles have a positive side and a negative side. For our purposes the positive side will be the shaded side in the diagrams. Flip the tiles so that the positive side of each tile is facing up. Trace one of each of the six tiles provided by your teacher on your paper. Leave plenty of space between each tracing.

   b. The dimensions of some of the tiles are shown at right. Label the dimensions of all the tiles next to the tracings you made.



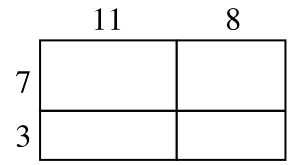
   c. The algebra tiles will be named according to each of their areas. Write the name of each tile in the center of your tracing with a colored pen or pencil. Make the name of the tile stand out.

   d. Below each tile write "P =" and then find the perimeter of each tile.

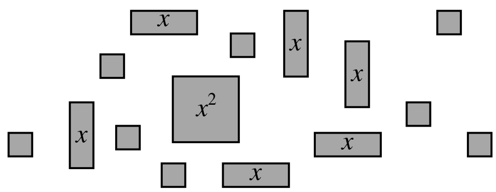
Exploring an Area Model

Algebra tiles can be used to represent algebraic equations. In this lesson, you will use algebra tiles to represent expressions using multiplication.

3. For the entire rectangle at right, find the area of each part and then find the area of the whole.



4. Your teacher will put this group of tiles on the overhead:



   a. Using your own tiles, arrange the same group of tiles into one large rectangle, with the x^2-tile in the lower left corner. On your paper, sketch what your rectangle looks like.

   b. What are the dimensions (length and width) of the rectangle you made? Label your sketch with its dimensions, then write the area of the rectangle as a *product*, that is, length · width.

   c. The area of a rectangle can also be written as the sum of the areas of all its parts. Write the area of the rectangle as the *sum* of its parts. Simplify your expression for the sum of the rectangle's parts.

   d. Write an equation that shows that the area written as a product is equivalent to the area written as a sum.

5. Your teacher will assign several of the expressions below. For each expression, build a rectangle using all of the tiles, if possible. Sketch each rectangle, find its dimensions, and write an expression showing the equivalence of the area as a *sum* (like $x^2 + 5x + 6$) and as a *product* (like $(x + 3)(x + 2)$). If it is not possible to build a rectangle, explain why not.

   a. $x^2 + 3x + 2$ b. $6x + 15$

   c. $2x^2 + 7x + 6$ d. $xy + x + y + 1$

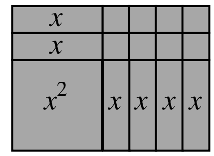
   e. $y^2 + xy + 2x + 2y$ f. $3x^2 + 4x + 1$

MATH NOTES

METHODS AND MEANINGS

Multiplying Algebraic Expressions with Tiles

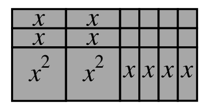
The area of a rectangle can be written two different ways. It can be written as a *product* of its width and length or as a *sum* of its parts. For example, the area of the shaded rectangle at right can be written two ways:



$$\underbrace{(x+4)}_{\textit{length}}\underbrace{(x+2)}_{\textit{width}} \quad = \quad \underbrace{x^2+6x+8}_{\textit{area}}$$

area as a product = area as a sum

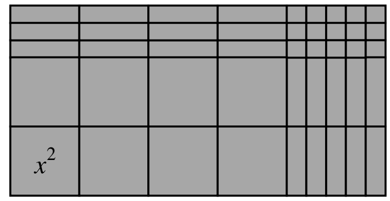
6. Write the area of the rectangle at right as a *product* and as a *sum*.



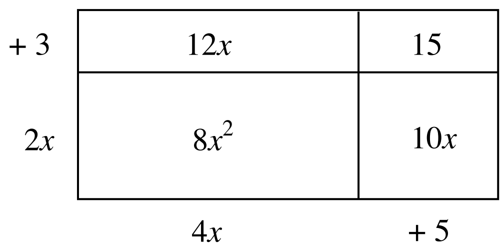
Using Generic Rectangles to Multiply

You have used algebra tiles and the concept of area to multiply polynomial expressions. In this lesson, you will be introduced to a tool that will help you find the product of the dimensions of a rectangle. This will allow you to multiply expressions without tiles.

7. Write the area as a *product* and as a *sum* for the rectangle shown at right.



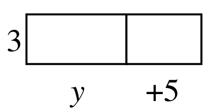
8. Now examine the following diagram. How is it similar to the set of tiles in problem 7? How is it different? Talk with your teammates and write down all of your observations.



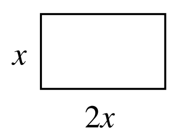
9. Diagrams like the one in problem 8 are referred to as **generic rectangles**. Generic rectangles allow you to use an area model to multiply expressions without using the algebra tiles. Using this model, you can multiply with values that are difficult to represent with tiles.

   Draw each of the following generic rectangles on your paper. Then find the area of each part and write the area of the whole rectangle as a *product* and as a *sum*.

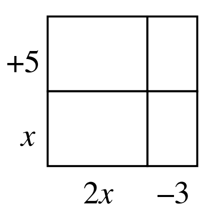
   a.



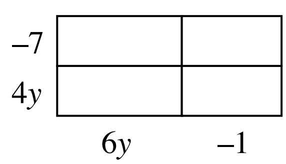
   b.



   c.



   d.



   e. How did you find the area of the individual parts of each generic rectangle?

10. Multiply and simplify the following expressions using either a generic rectangle or the Distributive Property. For part (a), verify that your solution is correct by building a rectangle with algebra tiles.

a. $(x + 5)(3x + 2)$

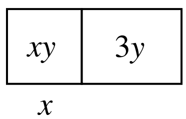
b. $(2y - 5)(5y + 7)$

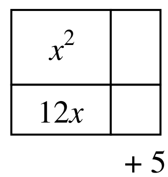
c. $3x(6x^2 - 11y)$

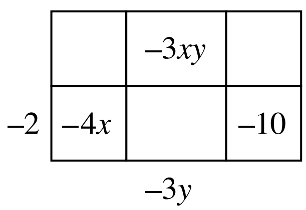
d. $(5w - 2p)(3w + p - 4)$

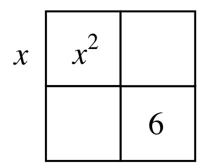
11. THE GENERIC RECTANGLE CHALLENGE

Copy each of the generic rectangles below and fill in the missing dimensions and areas. Then write the entire area as a product and as a sum. Be prepared to share your reasoning with the class.

a. 

b. 

c. 

d. 

Introduction to Factoring Quadratic Expressions

Now that you have learned how to multiply algebraic expressions using algebra tiles and generic rectangles, we will focus on reversing this process: How can you find a product when given a sum?

12. The process of changing a sum to a product is called **factoring**. Can every expression be factored? That is, *does every sum have a product that can be represented with tiles?*

    Investigate this question by building rectangles with algebra tiles for the following expressions. For each one, write the area as a sum and as a product. If you cannot build a rectangle, be prepared to convince the class that no rectangle exists (and thus the expression cannot be factored).

    a. $2x^2 + 7x + 6$

    b. $6x^2 + 7x + 2$

    c. $x^2 + 4x + 1$

    d. $2xy + 6x + y^2 + 3y$

13. Work with your team to find the sum and the product for the following generic rectangles. Are there any special strategies you discovered that can help you determine the dimensions of the rectangle? Be sure to share these strategies with your teammates.

    a.

$2x$	5
$6x^2$	$15x$

    b.

$-2y$	-6
$5xy$	$15x$

    c.

$-9x$	-12
$12x^2$	$16x$

14. While working on problem 13, Casey noticed a pattern with the diagonals of each generic rectangle. However, just before she shared her pattern with the rest of her team, she was called out of class! The drawing on her paper looked like the diagram below. Can you figure out what the two diagonals have in common?

$2x$	5
$6x^2$	$15x$

Factoring with Generic Rectangles

Since mathematics is often described as the study of patterns, it is not surprising that generic rectangles have many patterns. You saw one important pattern in the last lesson (Casey's pattern from problem 14). Now you will continue to use patterns while you develop a method to factor trinomial expressions.

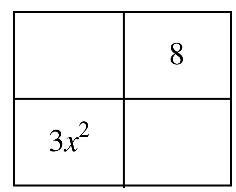
15. FACTORING QUADRATIC EXPRESSIONS

To develop a method for factoring without algebra tiles, first model how to factor with algebra tiles, and then look for connections within a generic rectangle.

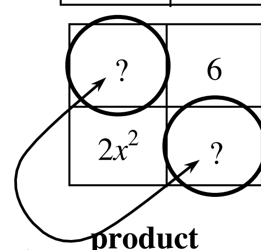
a. Using algebra tiles, factor $2x^2 + 5x + 3$; that is, use the tiles to build a rectangle, and then write its area as a product.

b. To factor with tiles (like you did in part (a)), you need to determine how to arrange the tiles to form a rectangle. Using a generic rectangle to factor requires a different process.

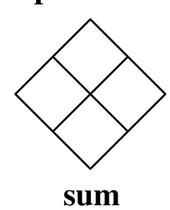
Miguel wants to use a generic rectangle to factor $3x^2 + 10x + 8$. He knows that $3x^2$ and 8 go into the rectangle in the locations shown at right. Finish the rectangle by deciding how to place the ten x-terms. Then write the area as a product.



c. Kelly wants to find a shortcut to factor $2x^2 + 7x + 6$. She knows that $2x^2$ and 6 go into the rectangle in the locations shown at right. She also remembers Casey's pattern for diagonals. Without actually factoring yet, what do you know about the missing two parts of the generic rectangle?

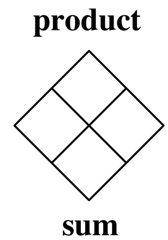


d. To complete Kelly's generic rectangle, you need two x-terms that have a sum of $7x$ and a product of $12x^2$. Create and solve a Diamond Problem that represents this situation.



e. Use your results from the Diamond Problem to complete the generic rectangle for $2x^2 + 7x + 6$, and then write the area as a product of factors.

16. Factoring with a generic rectangle is especially convenient when algebra tiles are not available or when the number of necessary tiles becomes too large to manage. Using a Diamond Problem helps avoid guessing and checking, which can at times be challenging. Use the process from problem 15 to factor $6x^2 + 17x + 12$. The questions below will guide your process.



a. When given a trinomial, such as $6x^2 + 17x + 12$, what two parts of a generic rectangle can you quickly complete?

b. How can you set up a Diamond Problem to help factor a trinomial such as $6x^2 + 17x + 12$? What goes on the top? What goes on the bottom?

c. Solve the Diamond Problem for $6x^2 + 17x + 12$ and complete its generic rectangle.

d. Write the area of the rectangle as a product.

17. Use the process you developed in problem 15 to factor the following quadratics, if possible. If a quadratic cannot be factored, justify your conclusion.

a. $x^2 + 9x + 18$

b. $4x^2 + 17x - 15$

c. $4x^2 - 8x + 3$

d. $3x^2 + 5x - 3$

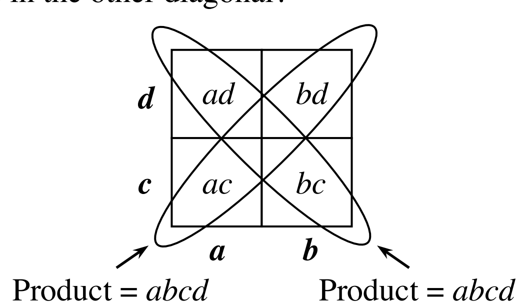
MATH NOTES

METHODS AND MEANINGS

Diagonals of Generic Rectangles

Why does Casey's pattern from problem 14 work? That is, why does the product of the terms in one diagonal of a 2-by-2 generic rectangle always equal the product of the terms in the other diagonal?

Examine the generic rectangle at right for $(a + b)(c + d)$. Notice that each of the resulting diagonals have a product of $abcd$. Thus, the product of the terms in the diagonals are equal.



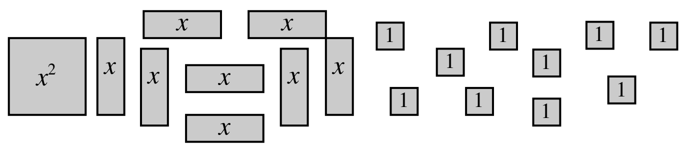
Completing the Square

Writing the equation of a parabola in graphing form, $f(x) = a(x - h)^2 + k$, made it easier to find the vertex and the x-intercept. But how can you change standard form, $f(x) = ax^2 + bx + c$, into graphing form? In this lesson, you will learn a new method called "completing the square."

18. COMPLETING THE SQUARE **eTool: bit.ly/2Qgwu9d**

Jessica was at home struggling with her Algebra homework. She had missed class and did not understand the new method called **completing the square**. She was supposed to use it to change $y = x^2 + 8x + 10$ to graphing form. Then her precocious younger sister, who was playing with algebra tiles, said, *"Hey, I bet I know what they mean."* Anita's Algebra class had been using tiles to multiply and factor binomials.

Anita explained: "$x^2 + 8x + 10$ *would look like this*;"



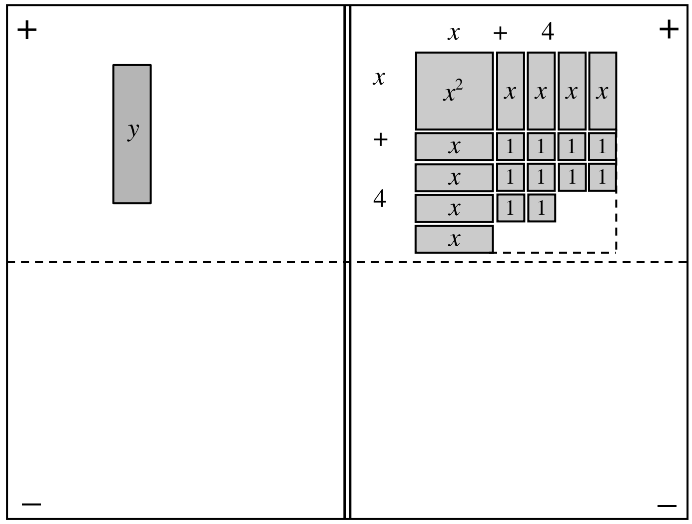
"Yes," said Jessica, *"I'm taking Algebra too, remember?"*

Anita continued, *"And you need to make it into a square!"*

Problem continues on next page. →

18. *Problem continued from previous page.*

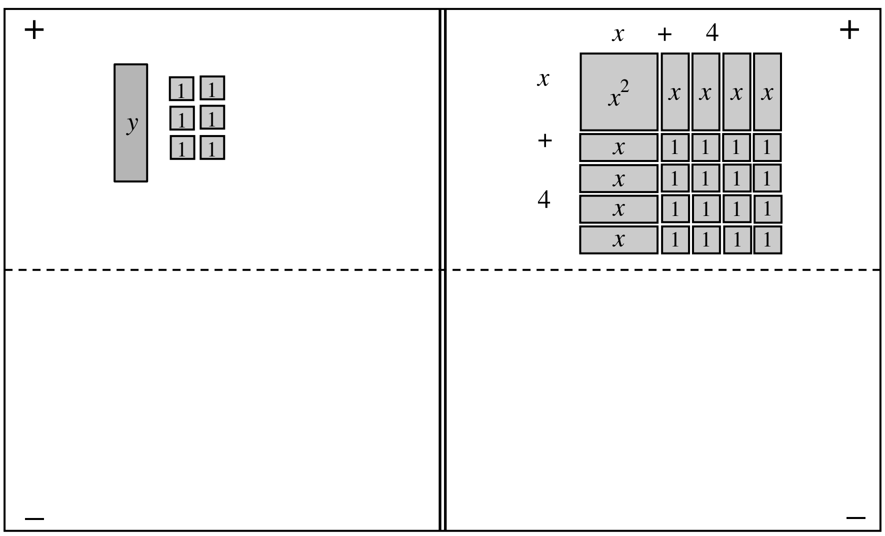
"OK," said Jessica, and she arranged her tiles on an equation mat as shown at right.



"Oh," said Jessica. *"I need 16 small unit tiles to fill in the corner!"*

"But you only have 10," Anita reminded her.

"Right, I only have ten," Jessica replied. She drew the outline of the whole square and said: *"Oh, I get it! To* ***complete the square****, I need to add six tiles to each side of the equation:"*



"Oh, I see," said Anita. *"You started with* $y = x^2 + 8x + 10$*, but now you can rewrite it as* $y + 6 = (x + 4)^2$*."*

"Thank you so much, Anita! Now I can easily write the function in graphing form, $y = (x + 4)^2 - 6$*."*

How can you use your graphing calculator to verify that $y = x^2 + 8x + 10$ and $y = (x + 4)^2 - 6$ are equivalent functions?

19. Write each function in graphing form, then state the vertex and y-intercept of each parabola.

a. $f(x) = x^2 + 6x + 7$

b. $f(x) = x^2 + 4x + 11$

20. How could you complete the square to change $f(x) = x^2 + 5x + 2$ into graphing form? How would you split the five x-tiles into two equal parts?

Jessica decided to use force! She cut one tile in half, as shown below. Then she added her two small unit tiles.

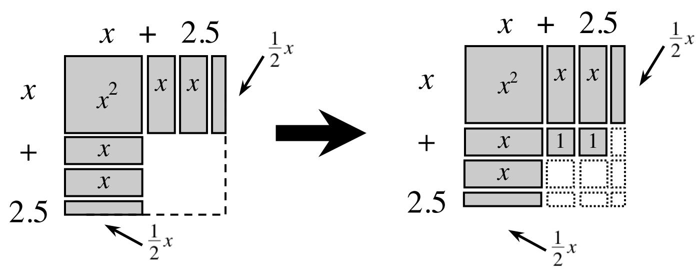


Figure A Figure B

a. How many small unit tiles are missing from Jessica's square?

b. Write the graphing form of the function, name the vertex and y-intercept, and sketch the graph.

Polynomial Division

When you graph polynomial functions, you know that the factored form of a polynomial is very useful for finding the roots of the function or the x-intercepts of the graph. But what happens when you do not have the factored form and you need to find all of the roots? You will investigate the answer to this question in this lesson.

21. Andre needs to find the exact roots of the function $f(x) = x^3 + 2x^2 - 7x - 2$. When he uses his graphing calculator, he can see that one of the x-intercepts is 2, but there are two other intercepts that he cannot identify exactly.

    Andre remembers that he learned how to multiply binomials and other polynomials using area models. He figures that since division is the inverse (or undo) operation for multiplication, he should be able to reverse the multiplication process to divide. As he thinks about that idea, he comes across the following news article.

Polydoku Craze Sweeping Nation!

(CPM) - Math enthusiasts around the nation have entered a new puzzle craze involving the multiplication of polynomials. The goal of the game, which enthusiasts have named Polydoku, is to fill in squares so that the multiplication of two polynomials will be completed.

	1	2	3	4	5
A	$\times$	$2x^3$	$-x^2$	$+3x$	-1
B	$3x$	$6x^4$	$-3x^3$	$9x^2$	$-3x$
C	-2	$-4x^3$	$2x^2$	$-6x$	2

$\mathbf{6x^4 \quad -7x^3 \quad +11x^2 \quad -9x \quad +2}$

The game shown at right, for example, represents the multiplication of $(3x-2)(2x^3 - x^2 + 3x - 1) = 6x^4 - 7x^3 + 11x^2 - 9x + 2$.

Most of the squares are blank at the start of the game. While the beginner level provides the factors (in the gray squares), some of the factors are missing in the more advanced levels.

What do you think Andre needs to be able to do to find the other roots?

22. Andre decided to join the craze and try some Polydoku puzzles, but he is not sure how to fill in some of the squares. Help him by answering parts (a) and (b) below about the Polydoku puzzle in the news article he read (found in problem 21), then complete part (c).

a. Explain how the term $2x^2$ in cell C3 of the news article was generated.

b. What values were combined to get $-7x^3$ in the news article answer?

c. Copy and complete the Polydoku puzzle at right.

	1	2	3	4	5
A	$\times$	$4x^3$	$+6x^2$	$-2x$	-5
B	$2x$				
C	-3				

23. Jessica is about to start the intermediate-level Polydoku puzzle shown at right. Show Jessica how to complete the puzzle. Make sure you can justify your solution.

	1	2	3	4
A	$\times$			
B	$2x$			
C	$+5$			
		$\mathbf{6x^3}$	$\mathbf{+7x^2}$	$\mathbf{-16x}$ $\mathbf{+10}$

Use your results to complete the statements below.

$$\frac{6x^3+7x^2-16x+10}{2x+5} = \_\_\_\_\_\_\_\_\_\_ \quad \text{and } (2x+5)\cdot \_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_$$

24. Unfortunately, Jessica made a mistake when she copied the problem. The constant term of the original polynomial was supposed to have the value + 18 (not + 10). She does not want to start all over again to solve the puzzle.

a. Jessica realizes that she would now have 8 remaining from the original expression. What is the significance of this 8?

b. Jessica writes her work as shown below:

$$\frac{6x^3+7x^2-16x+18}{2x+5} = \frac{(6x^3+7x^2-16x+10)+8}{2x+5} = 3x^2-4x+2 \text{ , remainder 8}$$

Gina thinks that there is a way to write the answer without using the word "remainder." Discuss this with your team and find another way to write the result. Be prepared to share your results and your reasoning with the class.

c. Use Jessica and Gina's method to divide $(6x^3 + 11x^2 - 12x - 1) \div (3x + 1)$.

25. Now work with your team to help Andre solve his original problem (problem 21). Find all of the roots (exact zeros) of the polynomial.

MATH NOTES

METHODS AND MEANINGS

Polynomial Division

The examples below show two methods for dividing $x^4 - 6x^3 + 18x - 1$ by $x - 2$. In both cases, the remainder is written as a fraction.

Using long division:

$$\begin{array}{r}
x^3 - 4x^2 - 8x + 2 \\
x - 2\overline{\smash{)}\,x^4 - 6x^3 + 0x^2 + 18x - 1} \\
\underline{x^4 - 2x^3} \qquad\qquad\qquad\quad \\
-4x^3 + 0x^2 \qquad\qquad\quad \\
\underline{-4x^3 + 8x^2} \qquad\qquad\quad \\
-8x^2 + 18x \qquad\quad \\
\underline{-8x^2 + 16x} \qquad\quad \\
2x - 1 \\
\underline{2x - 4} \\
3
\end{array}$$

Answer: $x^3 - 4x^2 - 8x + 2 + \frac{3}{x-2}$

Using area models:

	x^3	$-4x^2$	$-8x$	$+2$	Remainder
x	x^4	$-4x^3$	$-8x^2$	$+2x$	3
-2	$-2x^3$	$+8x^2$	$+16x$	-4	

$x^4 \quad -6x^3 \quad +0x^2 \quad +18x \quad -1$

Answer: $x^3 - 4x^2 - 8x + 2 + \frac{3}{x-2}$

Therefore, $(x^4 - 6x^3 + 18x - 1) \div (x - 2) = x^3 - 4x^2 - 8x + 2 + \frac{3}{x-2}$ and $(x - 2)(x^3 - 4x^2 - 8x + 2 + \frac{3}{x-2}) = x^4 - 6x^3 + 18x - 1$.

Mathematics | Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1 **Make sense of problems and persevere in solving them.**
- Find meaning in problems
- Look for entry points
- Analyze, conjecture and plan solution pathways
- Monitor and adjust
- Verify answers
- Ask themselves the question: "Does this make sense?"

2 **Reason abstractly and quantitatively.**
- Make sense of quantities and their relationships in problems
- Learn to contextualize and decontextualize
- Create coherent representations of problems

3 **Construct viable arguments and critique the reasoning of others.**
- Understand and use information to construct arguments
- Make and explore the truth of conjectures
- Recognize and use counterexamples
- Justify conclusions and respond to arguments of others

4 **Model with mathematics.**
- Apply mathematics to problems in everyday life
- Make assumptions and approximations to simplify a complicated situation
- Identify quantities in a practical situation
- Interpret results in the context of the situation and reflect on whether the results make sense

5 **Use appropriate tools strategically.**
- Consider the available tools when solving problems
- Are familiar with tools appropriate for their grade or course (pencil and paper, concrete models, ruler, protractor, calculator, spreadsheet, computer programs, digital content located on a website, and other technological tools)
- Make sound decisions of which of these tools might be helpful

6 **Attend to precision.**
- Communicate precisely to others
- Use clear definitions, state the meaning of symbols and are careful about specifying units of measure and labeling axes
- Calculate accurately and efficiently

7 **Look for and make use of structure.**
- Discern patterns and structures
- Can step back for an overview and shift perspective
- See complicated things as single objects or as being composed of several objects

8 **Look for and express regularity in repeated reasoning.**
- Notice if calculations are repeated and look both for general methods and shortcuts
- In solving problems, maintain oversight of the process while attending to detail
- Evaluate the reasonableness of their immediate results